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OPERATOR-VALUED CHANDRASEKHAR H-FUNCTIONS. (U)

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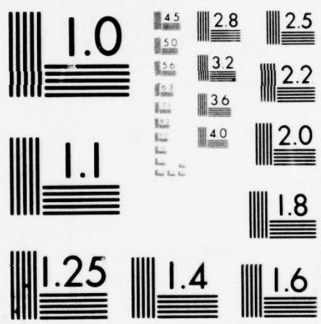

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OPERATOR-VALUED CHANDRASEKHAR H-FUNCTIONS

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See back page for 1473

December 1977

(Received August 5, 1977)

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OPERATOR-VALUED CHANDRASEKHAR H-FUNCTIONS

C. T. Kelley

Technical Summary Report # 1813
December 1977

ABSTRACT

Operator valued analogs of the Chandrasekhar H-function that occur in the study of neutron transport in a slab with continuous energy dependence and anisotropic scattering satisfy a system of nonlinear integral equations. An appropriate Banach space setting is found for the study of this system. We show that the system may be solved by iteration. We extend the domain of analyticity of H_r and H_ℓ by means of bifurcation theory.

AMS (MOS) Subject Classifications: 45E10, 45G99, 47G05, 82.45

Key Words: Chandrasekhar H-equation, Neutron transport theory, Bifurcation, Operator valued function

Work Unit Number 1 (Applied Analysis)

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Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and the National Science Foundation under Grant No. MCS75-17385 A01.

SIGNIFICANCE AND EXPLANATION

The Chandrasekhar H-equation is the following non-linear integral equation:

$$H(\mu) = 1 + \zeta H(\mu) \int_0^1 \frac{\mu}{\mu + \nu} H(\nu)\psi(\nu)d\nu ,$$

where ψ is given, ζ is a parameter, and H is the unknown. This equation is of importance in a variety of physical situations involving transfer of energy by radiative scattering processes, e.g. neutron transport.

The H-equation is related to convolution equations of the following type

$$(*) \qquad f(x) - \int_0^\infty k(x - y)f(y)dy = g(x) .$$

In anisotropic radiative transfer, the following generalization of (*) occurs

$$(**) \qquad f(x,\omega) - \int_0^\infty \int_\Omega k(x - y,\omega,\omega')f(\omega')d\omega' dy = g(x,\omega) .$$

The equation (**) is studied in this paper by treating it as an equation of the form (*). We must then interpret k as an operator, i.e.

$$k(x - y)f(y) = \int_\Omega k(x - y,\omega,\omega')f(y,\omega')d\omega' .$$

An explicit formula for k is given by equation (1.8) in the paper.

We consider here an H-equation in which H is regarded as an operator-valued function on $L_\infty(\Omega)$, i.e.

$$H(\mu)f = f(\omega) - \int_\Omega h(\mu,\omega,\omega')f(\omega')d\omega' .$$

This generalization appears in the study of radiative transfer in a slab with anisotropic scattering and continuous energy dependence. We show that we may solve our equation by iteration and obtain a bifurcation result, i.e. we show that in some cases there are two solutions. Numerical and asymptotic results of interest in transport theory may follow from our work.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

OPERATOR-VALUED CHANDRASEKHAR H-FUNCTIONS

C. T. Kelley

I. Introduction

In [10], Mullikin considered Wiener-Hopf factorizations of certain operator-valued functions that arise in the study of one-speed neutron transport in a slab with anisotropic scattering. This factorization has the form

$$(1.1) \qquad (I - \zeta\hat{K}(\lambda))\mathcal{H}_r(\lambda,\zeta)\mathcal{H}_\ell(-\lambda,\zeta) = I \ .$$

In (1.1) $\hat{K}$ is the Fourier transform of a compact operator-valued function $K(x)$ and I is the identity. $\mathcal{H}_r - I$ and $\mathcal{H}_\ell - I$ are Fourier transforms of compact operator-valued functions Γ_r and Γ_ℓ whose supports lie in $(0,\infty)$. ζ is a complex parameter and so that

$$(1.2) \qquad \sup_{\lambda \in \mathbb{R}} \|\hat{K}(\lambda)\|_{sp} = 1 \ .$$

In (1.2), $\|\cdot\|_{sp}$ denotes spectral radius. Mullikin derived a system of coupled nonlinear integral equations for the functions $H_r(\nu,\zeta)$ and $H_\ell(\nu,\zeta)$, where

$$(1.3) \qquad H_{r,\ell}(\nu,\zeta) = \mathcal{H}_{r,\ell}(i/\nu,\zeta), \ \mathrm{Re}\,\nu \geq 0 \ .$$

The purpose of this paper is three-fold. First we show that Mullikin's equations are valid with only minor modifications in the more general case of anisotopic transport depending continuously on energy. We develop notation that allows the nonlinear system for H_r and H_ℓ to be written in a compact way in a convenient Banach space setting. We then show that the equations for H_r and H_ℓ may be solved by an iterative method. Mullikin and the author [8] have recently proved a similar result in the case of isotropic scattering. Earlier results of this type may be found in [1], [2], and [6]. The proof given here differs from that of [8] in that the power series expansions of H_r and H_ℓ play a direct role.

Finally, we show that H_r and H_ℓ are analytic in $\sqrt{1-\zeta}$ for ζ in a cut neighborhood of 1. Our proof depends on a bifurcation analysis of the nonlinear system

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and the National Science Foundation under Grant No. MCS75-17385 A01.

for H_r and H_ℓ. Applications of results of this nature may be found in [7], [9], and [11].

The equation for steady state neutron transport in a slab with anisotropic scattering and continuous energy dependence may be written as follows, [12],

$$\mu \frac{\partial\psi}{\partial x}(x,\omega) + \sigma\psi(x,\omega) = F(x,\omega) \tag{1.4}$$

$$F(x,\omega) = S(x,\omega) + \zeta \int_\Omega K_0(\omega,\omega')\psi(x,\omega')d\omega' .$$

Here $\omega = (E,\vec{\varphi}) \in [E_0,E_1] \times S^2 = \Omega$, $x > 0$. E denotes energy and $\vec{\varphi}$ a direction vector μ is the cosine of the angle made by the vector $\vec{\varphi}$ and the positive x-axis σ is a continuous positive function of E on $[E_0,E_1]$, $\sigma \geq 1$. K is continuous on $\Omega \times \Omega$ and satisfies

(1.5) The integral operator K_0 on $C(\Omega)$ having kernel $K_0(\omega,\omega)\sigma^{-1}(E')$ is positive in the sense of [5] and has spectral radius 1.

$S(x,\omega)$ is continuous on $[0,\infty) \times \Omega$ and satisfies

$$\int_0^\infty \sup_{\omega\in\Omega} |S(x,\omega)|dx < \infty . \tag{1.6}$$

We seek solutions ψ that satisfy the boundary conditions

$$\psi(0,\omega) = 0, \quad \mu \geq 0 \tag{1.7}$$

$$\lim_{x\to\infty} \psi(x,\omega) = 0, \quad \mu \leq 0 .$$

We let $C(\Omega)$ and $L_\infty(\Omega)$ denote the spaces of continuous and essentially bounded functions on Ω with the sup-norm. $\mathbb{R}$ and $\mathbb{C}$ are the real and complex numbers.

For B a Banach space and I an interval let $L(B)$ be the algebra of bounded operators on B, $Com(B)$ the algebra of compact operators on B, $\mathbb{B}_p(I,B)$ $(1 \leq p < \infty)$ the space of Bochner integrable functions from I to B.

We define $K \in \mathbb{B}_1((0,\infty), Com(L_\infty(\Omega)))$ by

$$K(x)f(\omega) = \int_\Omega K_0(\omega,\omega')e^{-|x|\sigma(E')/|\mu'|}\,\theta(x\mu')\frac{1}{|\mu'|}\,f(\omega')d\omega' . \tag{1.8}$$

In (1.8), θ denotes the Heavyside function.

We may then convert the problem given by (1.4) and (1.7) to a Wiener-Hopf equation for $F \in \mathbb{B}_1((0,\infty), L_\infty(\Omega))$

$$F(x) - \zeta \int_0^\infty K(x-y)F(y)dy = S(x), \quad x > 0 . \tag{1.9}$$

The assumptions on K_0 and the theory developed in [4], [5] and [6] imply that (1.9) has a unique solution F for $|\zeta| < 1$ and that the function

$(I - \zeta\hat{K}(\lambda)) = I - \zeta \int_{-\infty}^{\infty} K(x)e^{ix\lambda}dx$, admits the Wiener-Hopf factorization (1.1). Let $\Omega_+ = \{\omega \in \Omega | \mu \geq 0\}$. We state some properties of H_r and H_ℓ that will be useful in what follows, these properties follow easily from [10].

Prop (1.1). Let $B(\lambda,\zeta)$ be any of the operators $H_{\ell,r}(\lambda,\zeta)$, $H^{-1}_{\ell,r}(\lambda,\zeta)$, then the following hold

(i) $B(\lambda,\zeta) - I$ is an analytic $\mathcal{C}om(L_\infty(\Omega))$ valued function for $|\zeta| < 1$, $\mathrm{Re}\lambda > 0$

(ii) $B(\lambda,\zeta)$ is a continuous $\mathcal{L}(L_\infty(\Omega))$-valued function for $|\zeta| < 1$, $\mathrm{Re}\lambda \geq 0$

(iii) $B(\lambda,\zeta) - I$ is an integral operator with kernel in $L_\infty(\Omega \times \Omega)$ for $|\zeta| < 1$ $\mathrm{Re}\lambda \geq 0$.

Moreover if $H_{r,\ell}(\nu,\zeta)$ is given by (1.3), the kernel $h_{r,\ell}(\nu,\omega,\omega',\zeta)$ of $H_{r,\ell}(\nu,\zeta) - I$ is nonnegative on $\Omega \times \Omega$, in fact

(iv) $h_\ell(\nu,\omega,\omega',\zeta) > 0$ on $\Omega \times \Omega$ $\nu \geq 0$, $0 < \zeta < 1$

(v) $h_r(\nu,\omega,\omega',\zeta) > 0$ on $\Omega \times \Omega_+$ $\nu \geq 0$, $0 < \zeta < 1$

(vi) h_ℓ is continuous on $[0,\infty) \times \Omega \times \Omega$ for $|\zeta| < 1$

(vii) h_r is continuous on $[0,\infty) \times \Omega \times \Omega_+$ for $|\zeta| < 1$

(viii) h_r and h_ℓ are increasing functions of ν for $0 < \zeta < 1$, and of ζ for $\nu \geq 0$, $0 \leq \zeta < 1$.

II.

If A is an integral operator with kernel in $L_\infty(\Omega \times \Omega)$, we denote by A' that integral operator having as its kernel the transpose kernel of A. We define $I' = I$.

We let N denote the algebra of integral operators on $C(\Omega)$ having kernels that are in $C(\Omega \times \Omega)$. We let $k_A(\omega,\omega')$ be the kernel of $A \in N$. We define

$$\|A\|_N = \sup_{(\omega,\omega')\in\Omega\times\Omega} |k_A(\omega,\omega')| . \tag{2.1}$$

N is a Banach algebra with this norm. Note that, for $A,B \in N$

$$k_{AB}(\omega,\omega') = \int_\Omega k_A(\omega,\omega_0)k_B(\omega_0,\omega)d\omega_0 . \tag{2.2}$$

We let N_+ denote the algebra of integral operators on $L_\infty(\Omega)$ having kernels which are continuous on $\Omega_+ \times \Omega$. The norm on N_+ is defined by

$$\|A\|_{N_+} = \sup_{\substack{\omega\in\Omega_+ \\ \omega'\in\Omega}} |(k_A(\omega,\omega')| . \tag{2.3}$$

As in [5], [6] and [7] we consider 2×2 diagonal matrices of the form

$$A = \begin{pmatrix} A_\ell & 0 \\ 0 & A_r' \end{pmatrix}, \tag{2.4}$$

where A_ℓ and A_r are integral operators having kernels in $L_\infty(\Omega \times \Omega)$. We denote the space of such matrices by X and define, for $A \in X$

$$\|A\|_X = \max(\|k_{A_\ell}\|_\infty, \|k_{A_r}\|_\infty) . \tag{2.5}$$

If $A,B \in X$, define AB and A^*

$$AB = \begin{pmatrix} A_\ell B_\ell & 0 \\ 0 & A_r'B_r' \end{pmatrix} \tag{2.6}$$

$$A^* = \begin{pmatrix} A_r & 0 \\ 0 & A_\ell' \end{pmatrix} . \tag{2.7}$$

Let X_0 be the algebra formed by adjoining the identity to X. We have

$$I_{X_0} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = I^*_{X_0} , \tag{2.8}$$

$$\| I_{X_0} \|_{X_0} = 1 . \tag{2.9}$$

When no confusion can result we drop the subscript X_0 and refer to I_{X_0} as I. Let $C([0,1),X_0) = C$ be the space of continuous X_0 valued functions. The norm on C is defined, for $F \in C$, by

$$\|F\|_C = \sup_{0 \le \nu \le 1} \|F(\nu)\|_{X_0} . \tag{2.10}$$

Define a map $L : C \to C([0,1],X)$ as follows. For $F(\nu) \in C$ write $F(\nu) = \alpha(\nu)I + G(\nu)$ where $G(\nu) \in C([0,1),X)$, then

$$k_{(LF)_\ell}(\nu) = K_0(\omega,\omega')\theta(-\mu') \frac{\nu}{\sigma(E')\nu + |\mu'|} \alpha(|\mu'|) \tag{2.11}$$

$$+ \int_\Omega K_0(\omega,\omega_0)\theta(-\mu_0) \frac{\nu}{\sigma(E_0)\nu + |\mu_0|} k_{G_\ell}\left(\frac{|\mu_0|}{\sigma(E_0)} \omega_0,\omega'\right) d\omega_0 ,$$

$$k_{(LF)'_r}(\nu,\omega,\omega') = \alpha(\mu)K'_0(\omega,\omega')\theta(\mu) \frac{\nu}{\sigma(E)\nu + |\mu|} \tag{2.12}$$

$$+ \int_\Omega K'_0(\omega,\omega_0) \frac{\nu}{\sigma(E)\nu + \mu} \theta(\mu) k_{G'_r}\left(\frac{|\mu|}{\sigma(E)} , \omega_0,\omega'\right) d\omega_0 .$$

Define $H(\nu,\zeta) = \begin{pmatrix} H_\ell(\nu,\zeta) & 0 \\ 0 & H'_r(\nu,\zeta) \end{pmatrix}$. We have, as in [10], for $|\zeta| < 1$, $\mathrm{Re}\nu \geq 0$,

$$H(\nu,\zeta) = I + \zeta L(H^*)(\nu,\zeta)H(\nu,\zeta) . \tag{2.13}$$

We consider this equation in the subspace C_0 of C given by

$$C_0 = \{F(\nu) = \alpha I + G(\nu) \in C | G_\ell \in N,\ G'_r \in N_+ \text{ for all } \nu,\ \alpha \in \mathbb{C}\} . \tag{2.14}$$

We may then consider (2.13) as an equation in the space C_0. We define $T \in X$ by

$$T = \begin{pmatrix} \hat{K}(0) & 0 \\ 0 & \hat{K}'(0) \end{pmatrix} . \tag{2.15}$$

The assumptions on K imply that if T is considered as an operator on $C(\Omega) \times C(\Omega)$ then there are positive functions u_r, u_ℓ in $C(\Omega)$ so that the vector $u = \begin{pmatrix} u_r \\ u_\ell \end{pmatrix}$ satisfies

(2.16) $$u - Tu = 0, \quad \int_\Omega u_r(\omega)u_\ell(\omega)d\omega = 1 .$$

If $F_1, F_2 \in C([0,1],X)$ we say $\alpha I + F_1 \geq \beta I + F_2$ if $\alpha \geq \beta$ and $k_{F_1} \geq k_{F_2}$ almost everywhere in $[0,1 \times \Omega \times \Omega$. If $u = \begin{pmatrix} u_r \\ u_\ell \end{pmatrix}$, $v = \begin{pmatrix} v_r \\ v_\ell \end{pmatrix} \in C(\Omega) \times C(\Omega)$ we say $u \geq v$ if $u_r \geq v_r$, $u_\ell \geq v_\ell$ everywhere in Ω. For $u \in C(\Omega) \times C(\Omega)$ and $\alpha \in \mathbb{R}$, we say $u \geq \alpha$ if $u \geq \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$.

Define $H(\zeta) \in X_0$ by

(2.17) $$H(\zeta) = \begin{pmatrix} H_\ell(0,\zeta) & 0 \\ 0 & H_r'(0,\zeta) \end{pmatrix} .$$

Note that

(2.18) $$H^*(\zeta)H(\zeta)(I - \zeta T) = I, \quad |\zeta| < 1 .$$

The following lemma is crucial to what follows. The proof is similar to that in [7].

<u>Lemma (2.1)</u>. The limit

$$\lim_{\zeta \to 1^-} H(\zeta) = H(1)$$

exists in C_0. Moreover, $H(1)$ satisfies

(2.19) $$H(1) = I + L(H^*(1))H(1) .$$

<u>Proof</u>. The second statement clearly follows from the first.

Define a map $L_0 \in L(C_0)$ by

(2.20) $$L_0 F = \lim_{\nu \to \infty} (LF(\nu)) .$$

The definition of H and (2.18) imply

$$H^{*-1}(\zeta) = I - \zeta(L_0(H^*))^* = H(\zeta)(I - \zeta T) .$$

For $0 < \zeta < 1$ we apply both sides of the above to the vector u and note that $H(\zeta) \geq I$ to get

$$u - \zeta(L_0(H^*))^* u \geq (1 - \zeta)u .$$

Set $M = L_0 - L$, we have

$$H^*(\nu,\zeta)(I - \zeta(L_0(H^*))^*) = I - \zeta H^*(\nu,\zeta)(M(H^*))^* .$$

Hence, as $H^*(\nu,\zeta) \geq 0$,

$$u - H^*(\nu,\zeta)(M(H^*))^* u \geq (1-\zeta)H^*(\nu,\zeta)u$$

or

$$H^*(\nu,\zeta)[(1-\zeta)u + \zeta(M(H^*))^* u] \leq u \ .$$

Let $v = (1-\zeta)u + \zeta(M(H^*))^* u = \begin{pmatrix} v_r \\ v_\ell \end{pmatrix}$. By the definition of M, $\inf_{\omega\in\Omega} v_\ell(\omega) = \alpha_\ell > 0$, and $\inf_{\omega\in\Omega_+} v_r(\omega) = \alpha_r > 0$, therefore h_r and h_ℓ are bounded on $\Omega \times \Omega \times [0,1]$ uniformly in ζ.

As $0 \leq H(\nu,\zeta) \leq H(\nu,\zeta_0)$ for $0 \leq \zeta \leq \zeta_0 < 1$, we have that $\lim_{\zeta\to 1^-} h'_{\ell,r}(\nu,\omega,\omega',\zeta) = h_{\ell,r}(\nu,\omega,\omega')$ exists in $L_1([0,1] \times \Omega \times \Omega)$ by the dominated convergence theorem.

To complete the proof one may now proceed exactly as in [7].

The lemma, implies that $H(\zeta)$ is analytic in $|\zeta| < 1$ and continuous on $|\zeta| \leq 1$.

The main result of this section is the following

<u>Theorem (2.1)</u>. The sequence H_n given for $|\zeta| \leq 1$

(2.21)
$$H_0 = I$$
$$H_{n+1} = I + \zeta L(H_n^*)H_n$$

converges to $H(\zeta)$ in C_0 uniformly for $|\zeta| \leq 1$.

<u>Proof</u>. $H(\zeta)$ is the unique solution of (2.13) which is analytic in $|\zeta| < 1$. Let $\sum_{n=0}^{\infty} \zeta^n P_n$ be the power series expansion of $H(\zeta)$ about $\zeta = 0$. We have that $P_n \geq 0$ for each n, $P_0 = I$, and, for $n \geq 1$, $P_n = \sum_{\ell+k=n} L(P_\ell^*)P_k$. Let $S_N = \sum_{n=0}^{N} P_n$. By considering the coefficients of the polynomials H_n, one can easily show

$$\|H(\zeta) - H_n(\zeta)\| \leq \|H(|\zeta|) - H_n(|\zeta|)\| \leq \|H(1) - H_n(1)\| \leq \|H(1) - S_n\| \ .$$

S_n converges to $H(1)$ by lemma (2.1), and the fact that $P_n \geq 0$ for each n. Hence, the proof is complete.

III.

In this section we show that $H(\nu,\zeta)$ is an analytic C_0 valued function of $\sqrt{1-\zeta}$ in a cut neighborhood of $\zeta = 1$. We do this by showing that $\zeta = 1$ is a *branch point* of order 2 of the equation (2.13). The proof follows the general theme of those in [7], [9], and [11].

In this section set $H = H(1)$, $\varepsilon = 1 - \zeta$, and $G(\varepsilon) = (H - H(\zeta))H^{-1}$. It is easy to show that G satisfies, for $\varepsilon \geq 0$

$$(3.1) \qquad G(\varepsilon) - HL((G(\varepsilon)H)^*) = \frac{\varepsilon}{1-\varepsilon}[(I - G(\varepsilon))H - I] - G(\varepsilon)HL((G(\varepsilon)H)^*) .$$

The Frechet derivative of the map given by (3.1) with respect to G at $(G,\varepsilon) = (0,0)$ is $(I - \mathcal{L})$, where $\mathcal{L} \in L(C_0)$ is given by

$$(3.2) \qquad \mathcal{L}(K) = HL((KH)^*), \quad K \in C_0 .$$

As in [7], [9], and [11] we consider an alternate operator $\mathcal{M}$ defined by

$$(3.3) \qquad \mathcal{M}(K) = HM((KH)^*), \quad K \in C_0 .$$

In (3.3) $M = L_0 - L$. We require

<u>Lemma (3.1)</u>. There is $P \in C_0$ satisfying

(i) $P^* = P$

(ii) P is independent of ν

(iii) $P \geq 0$

(iv) $P - \mathcal{M}(P) = 0$

(v) $P^2 = P$.

For the present we assume the lemma. A computation will show that the function $Q(\nu) = \nu P$ satisfies $(I - \mathcal{L})Q = 0$. Moreover if K is such that $(I - \mathcal{M})K = 0$, then the function $S(\nu) = \nu K$ satisfies $(I - \mathcal{L})S = 0$. Bifurcation of (3.1) at $\varepsilon = 0$ will take place if we can show that the only solutions to $(I - \mathcal{M})K = 0$ are of the form $K = \alpha P$, $\alpha \in \mathbb{C}$ and that the range of $(I - \mathcal{M})$ has codimension 1 in C_0.

For $K = \begin{pmatrix} K_\ell & 0 \\ 0 & K_r' \end{pmatrix}$ the operator $\mathcal{M}^2 K$ has the form

$$(3.4)\qquad M^2K = \begin{pmatrix} M_\ell^2 K_\ell & 0 \\ 0 & (M_r^2 K_r)' \end{pmatrix}$$

Moreover $M_\ell^2 K_\ell$ and $(M_r^2 K_r)'$ are strictly positive elments of $C([0,1],N)$ and $C([0,1],N_+)$ respectively if $K_\ell \geq 0$, $K_\ell \neq 0$ and $K_r \geq 0$, $K_r \neq 0$. As M^2 is compact on C_0, we have that, up to a constant multiple, P is the only solution to $(I - M^2)P = 0$ and hence to $(I - M)P = 0$ and that the range of $(I - M)$ has codimension 1 in C_0. Therefore $Q(\nu) = \nu P$ is, up to a constant multiple, the unique solution to $(I - L)Q = 0$. Moreover, there is $\Lambda \in C_0'$ such that $\Lambda((I - L)K) = 0$ for all $K \in C_0$ and $\Lambda(K) > 0$ if $K \geq 0$ and $K \neq 0$. One may then proceed as in [7], [9] and [11] to obtain

<u>Theorem (3.2)</u>. $H(\zeta)$ is analytic in $\sqrt{1-\zeta}$ for S sufficiently near 1, $\arg(1-\zeta) \neq \pi$. Moreover we have the expansion

$$(3.5)\qquad H(\zeta) = H - \alpha\varepsilon^{1/2} QH + O(\varepsilon) ,$$

where $Q(\nu) = \nu P$, and P is as in lemma (3.1),

$$(I - L)Q = 0 \quad \text{and} \quad \alpha = \left(\frac{\Lambda(H - I)}{\Lambda(\nu Q(\nu))}\right)^{1/2} > 0 .$$

In [7], [9], and [11], the functional Λ had a simple integral representation. In this case, however, it is not clear that Λ can be written in a simple way.

It remains to prove lemma (3.1). Let u be given by (2.16). Consider the equality

$$(3.6)\qquad (1 - \zeta)H(0,\zeta)u = (I - \zeta(L_0(H^*(\zeta))^*)u .$$

The right hand side of (3.6) has a limit as ζ approaches 1, hence

$$(3.7)\qquad \lim_{\substack{\zeta\to 1 \\ |\zeta|<1}} (1 - \zeta)H(0,\zeta)u = w \geq 0 .$$

If $w = 0$, then $(I - (L_0(H^*))^*)u = 0$ and hence there is $v = \begin{pmatrix} v_r \\ v_\ell \end{pmatrix}$, $v \neq 0$ such that

$$(3.8)\qquad v - L_0(H^*)v = 0 \quad \text{and} \quad \int v_r(\omega)v_\ell(\omega)d\omega = 1 .$$

Define, for $x = \begin{pmatrix} x_r \\ x_\ell \end{pmatrix} \in C(\Omega) \times C(\Omega)$

$$(3.9) \qquad Px = \begin{pmatrix} (x_r, v_\ell) v_r \\ (x_\ell, v_r) v_\ell \end{pmatrix} .$$

Then P satisfies (i)-(v). If $\omega \neq 0$ we have, by (2.13),

$$(1 - \zeta)H(0,\zeta)u = (1 - \zeta)u + \zeta L_0(H^*(\zeta))(1 - \zeta)H(0,\zeta)u .$$

Hence $w = L(H^*)w$. Therefore v satisfying (3.8) exists and P may be defined by (3.9).

Theorem (3.2) implies, in fact, that $w = 0$, and hence P is unique and given by (3.8) and (3.9).

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 14 MRC-TSR-1813	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER 9 Technical Summary rept.
4. TITLE (and Subtitle) 6 OPERATOR-VALUED CHANDRASEKHAR H-FUNCTIONS.		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 C. T. Kelley		8. CONTRACT OR GRANT NUMBER(s) 15 DAAG29-75-C-0024, NSF-MCS75-17385 [redacted]
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1 (Applied Analysis)
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below.		12. REPORT DATE 11 Dec[redacted]77 12 15p. 13. NUMBER OF PAGES 11
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
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National Science Foundation
Washington, D.C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Chandrasekhar H-equation
Neutron transport theory
Bifurcation
Operator valued function

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Operator valued analogs of the Chandrasekhar H-function that occur in the study of neutron transport in a slab with continuous energy dependence and anisotropic scattering satisfy a system of nonlinear integral equations. An appropriate Banach space setting is found for the study of this system. ~~We show that~~ the system may be solved by iteration. ~~We extend~~ the domain of analyticity of H_r and H_ℓ is extended by means of bifurcation theory.

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